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CLOSED-FORM SOLUTIONS AND CONSERVED VECTORS OF THE (3+1)-DIMENSIONAL NEGATIVE-ORDER KdV EQUATION

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Abstract. In this paper, we study the (3+1)-dimensional negative-order Korteweg-de Vries equation, which was recently formulated by using the recursion operator of the Korteweg-de Vries equation. Firstly, we eliminate the integral appearing in the equation and use the Lie symmetries of the resultant partial differential equation to perform symmetry reductions. The obtained fourth-order ordinary differential equation is then solved by two methods and closed-form solutions are constructed. In addition, we compute the conserved vectors of the underlying equation by engaging Noether's theorem.

Keywords: (3+1)-dimensional negative-order KdV equation, Lie point symmetries, exact solution, Kudryashov method, conservation laws.

AMS Subject Classification: 35A09.

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1 Introduction

It is well-known that the dynamics of several physical systems are governed by nonlinear partial differential equations (NPDEs) and as such have great importance to our contemporary world. In the past few decades there has been immense prominence in understanding and modeling nonlinear processes that are governed by NPDEs in many fields of thriving importance such as nonlinear dynamics, fluids and continuum mechanics, meteorology and climate, astronomy, oceanography, system theory and control, operations research, to name but a few. Due to its applications in real-world problems, NPDEs have become one of the most vital areas of research in the modern era of mathematical research (See for example reference list).

Notwithstanding the significance of obtaining the closed-form solutions of NPDEs, there is still the formidable problem of determining new methods to invent new closed-form or approximate solutions. In this regard, researchers have recently established various special methods for finding closed-form solutions of NPDEs, which include the inverse scattering transform method (Ablowitz et al., 1991), bifurcation method (Zhang & Khalique, 2018), the simplest equation method (Kudryashov, 2005), the extended simplest equation method (Kudryashov, 2008), Kudryashov method (Kudryashov, 2012), Hirota method (Hirota, 2004), Bäcklund transformation (Gu, 1990), Darboux transformation (Matveev & Salle, 1991), the homogeneous balance method (Wang et al., 1996), $(G'/G)-$expansion method (Wang et al., 2005), Lie symmetry method (Ovsiannikov, 1982; Bluman & Kumei, 1989; Olver, 1993; Ibragimov, 1995, 1999) and so on.

Since the middle of the nineteenth century, Lie symmetry analysis, which was pioneered by Sophus Lie (1842-1899), the Norwegian mathematician, has demonstrated the fact that it is one of the most effective and powerful techniques for obtaining closed-form solutions to NPDEs. See for example (Ovsiannikov, 1982; Bluman & Kumei, 1989; Olver, 1993; Ibragimov, 1995, 1999; Motsepa et al., 2017; Khalique & Adem, 2018; Khalique et al., 2018; Khalique & Moleleki, 2019; Khalique et al., 2019). E. Galois (1811-1832) had used group theory to solve the algebraic equations and this inspired Lie. He realized that the hotchpotch methods for solving the ordinary differential equations (ODEs) could be consolidated and so he embarked on a study to solve differential equations analogous to solving algebraic equations.

Conservation laws play a pivotal role in the investigation of partial differential equations (PDEs). They provide basic conserved physical quantities, like conservation of angular momentum, conservation of energy, etc. One can make use of conservation laws to identify whether a PDE is completely integrable. They can also be used in checking the validity of numerical solution methods. Recently, exact solutions were obtained for some PDEs using conservation laws (Noether, 1918; Bluman et al., 2010; Leveque, 1992; Ibragimov, 2007; Naz et al., 2008; Sjöberg, 2009; Yasar & Özer, 2011; Sarlet, 2010; Motsepa et al., 2018; Anco, 2017; Khalique & Abdallah, 2020; Bruzón & Gandarias, 2018). The celebrated Noether's theorem Noether (1918) furnishes an ingenious and effective way of determining conservation laws. It provides a formula for determining local conservation laws once a Noether symmetry connected to a Lagrangian is established for an Euler-Lagrange equation.

The celebrated Korteweg-de Vries (KdV) equation given by

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1}$$

is the result of research concerning long waves in shallow water surfaces. Here t and x denote time and position, respectively and $u(x, t)$ represents the wave surface. It was first introduced by Boussinesq (1877) and rediscovered in 1895 by Darrigol (2005); De Jager (2006). In Wazwaz (2017b) the author, using the techniques given in Olver (1977); Zhang et al. (2009); Gurses (2013), derived several (3+1)-dimensional negative-order KdV equations.

In this paper, we study one such equation, namely, the (3 + 1)-dimensional negative-order KdV equation, model II, given by

$$u_x + u_y - 4u_tu + 4u_zu + 2u_x\partial_x^{-1}u_z - 2u_x\partial_x^{-1}u_t + u_{xxt} - u_{xxz} = 0. \tag{2}$$

The structure of the paper is as follows: In Section 2, we employ the Lie symmetry method together with Kudryashov method to find closed-form solutions of (2). Thereafter, in Section 3, we invoke Noether's theorem and derive its conservation laws. Finally, concluding remarks are given in Section 4.

2 Exact solutions of (2)

In this section, we determine exact solutions of the (3+1)-dimensional negative-order KdV equation, model II (2). However, first we get rid of the two integral terms in the equation and obtain a fourth-order PDE. Using Lie symmetries of this fourth-order PDE, we perform symmetry reductions and transform it into a fourth-order nonlinear ordinary differential equation (ODE). Firstly, we carry out direct integration of the ODE and obtain exact solutions of (2). Secondly, we invoke Kudryashov method and compute more exact solutions of (2).

2.1 Lie symmetries and symmetry reductions

We make use of the transformation

$$u(x, y, z, t) = v_x(x, y, z, t)$$

and eliminate the integrals appearing in the equation. Then equation (2) becomes

$$v_{xx} + v_{xy} - 4v_x v_{xt} - 2v_t v_{xx} + 4v_x v_{xz} + 2v_{xx} v_z + v_{xxxt} - v_{xxxz} = 0. \tag{3}$$

Consider the symmetry group of (3) brought about by the vector field

$$\Gamma = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \xi^3 \frac{\partial}{\partial z} + \xi^4 \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial v},$$

where ξ^i, $i = 1, 2, 3, 4$ and η depend on variables x, y, z, t and v. Applying the fourth prolongation $\mathrm{pr}^{(4)}\Gamma$ to equation (3), expanding and splitting on derivatives of v, we obtain an overdetermined system of twenty two linear homogeneous partial differential equations

$$\begin{aligned}
&{\xi^1}_{xx} = 0, \eta_{xx} = 0, \eta_{xy} = 0, {\xi^1}_{xv} = 0, {\xi^4}_x = 0, {\xi^3}_x = 0, {\xi^2}_x = 0, {\xi^2}_{yv} = 0,\\
&{\xi^4}_v = 0, {\xi^3}_v = 0, {\xi^2}_v = 0, {\xi^1}_v = 0, \eta_v + {\xi^1}_x = 0, {\xi^1}_{xt} - {\xi^1}_{xz} = 0,\\
&4\eta_x + {\xi^4}_y = 0, {\xi^3}_y - 4\eta_x = 0, {\xi^2}_t - {\xi^2}_z = 0, {\xi^1}_t - {\xi^1}_z = 0,\\
&2\eta_{xt} + {\xi^1}_{xy} - 2\eta_{xz} = 0, {\xi^3}_t - 2{\xi^1}_x + {\xi^2}_y - {\xi^3}_z = 0,\\
&{\xi^4}_t + 2{\xi^1}_x - {\xi^2}_y - {\xi^4}_z = 0, 2\eta_t + {\xi^1}_x - {\xi^2}_y + {\xi^1}_y - 2\eta_z = 0.
\end{aligned}$$

Solving this system of PDEs one obtains the following nine Lie point symmetries:

$$\begin{aligned}
\Gamma_1 =& F^1(t+z)\frac{\partial}{\partial y},\ \Gamma_2 = F^2(t+z)\frac{\partial}{\partial t},\ \Gamma_3 = F^3(y, t+z)\frac{\partial}{\partial v},\\
\Gamma_4 =& F^4(t+z)\frac{\partial}{\partial z} - F^4(t+z)\frac{\partial}{\partial t},\ \Gamma_5 = 2F^5(y, t+z)\frac{\partial}{\partial x} + z{F^5}_y\frac{\partial}{\partial v},\\
\Gamma_6 =& 4yF^6(t+z)\frac{\partial}{\partial t} - xF^6(t+z)\frac{\partial}{\partial v} - 4yF^6(t+z)\frac{\partial}{\partial z},\\
\Gamma_7 =& \left\{-2vF^7(t+z) - zF^7(t+z)\right\}\frac{\partial}{\partial v} + 2xF^7(t+z)\frac{\partial}{\partial x} + 4yF^7(t+z)\frac{\partial}{\partial y},\\
\Gamma_8 =& -zF^8(t+z)\frac{\partial}{\partial v} - 2zF^8(t+z)\frac{\partial}{\partial t} + 2yF^8(t+z)\frac{\partial}{\partial y} + 2zF^8(t+z)\frac{\partial}{\partial z},\\
\Gamma_9 =& \left\{2vyF^9(t+z) - xzF^9(t+z) + 3yzF^9(t+z)\right\}\frac{\partial}{\partial v} + 4yzF^9(t+z)\frac{\partial}{\partial t},\\
&- 2xyF^9(t+z)\frac{\partial}{\partial x} - 4y^2F^9(t+z)\frac{\partial}{\partial y} - 4yzF^9(t+z)\frac{\partial}{\partial z}.
\end{aligned}$$

Taking the functions $F^m, m = 1, 2, \cdots, 9$ to be equal to 1, we obtain the following symmetries for equation (3):

$$\begin{aligned}
&X_1 = \frac{\partial}{\partial y},\ X_2 = \frac{\partial}{\partial t},\ X_3 = \frac{\partial}{\partial v},\ X_4 = \frac{\partial}{\partial z} - \frac{\partial}{\partial t},\ X_5 = \frac{\partial}{\partial x},\\
&X_6 = -4y\frac{\partial}{\partial z} + 4y\frac{\partial}{\partial t} - x\frac{\partial}{\partial v},\\
&X_7 = 2x\frac{\partial}{\partial x} + 4y\frac{\partial}{\partial y} - (2v+z)\frac{\partial}{\partial v},\\
&X_8 = 2y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z} - 2z\frac{\partial}{\partial t} - z\frac{\partial}{\partial v},\\
&X_9 = -2xy\frac{\partial}{\partial x} - 4y^2\frac{\partial}{\partial y} - 4yz\frac{\partial}{\partial z} + 4yz\frac{\partial}{\partial t} + (2vy - xz + 3yz)\frac{\partial}{\partial v}.
\end{aligned} \tag{4}$$

Consider the symmetry $X = \alpha X_1 + X_4 + X_5$, where α is a constant. Using this symmetry X, we reduce (3) to a PDE with one less independent variable, that is, three independent variables. The associated Lagrange system for X, on solving, gives the four invariants

$$f = \alpha x - y,\ g = \alpha z - y,\ h = \alpha t + y,\ \theta = v. \tag{5}$$

We now treat θ as a new dependent variable and f, g and h as new independent variables. The equation (3) then reduces to

$$\begin{aligned} &\alpha\theta_{ff} - \theta_{ff} - \alpha^3\theta_{fffg} + 4\alpha^2\theta_f\theta_{fg} + 2\alpha^2\theta_{ff}\theta_g - \theta_{fg} \\ &+ \alpha^3\theta_{fffh} - 4\alpha^2\theta_f\theta_{fh} - 2\alpha^2\theta_{ff}\theta_h + \theta_{fh} = 0. \end{aligned} \tag{6}$$

We now use the Lie symmetries of (6) and reduce it to a PDE in two independent variables. Equation (6) has

$$R_1 = \frac{\partial}{\partial g},\ R_2 = \frac{\partial}{\partial f},\ R_3 = \frac{\partial}{\partial h} - \frac{\partial}{\partial g}\ R_4 = \frac{\partial}{\partial \theta}$$

as its symmetries. The symmetry $R = \beta R_2 + R_3$, where β is a constant, will reduce (6) to a PDE in two independent variables. Solving the associated Lagrange system for R, yields the three invariants

$$r = f + \beta g,\ s = f - \beta h,\ \phi = \theta. \tag{7}$$

We now treat ϕ as new dependent variable and r and s as new independent variables. Doing this, the PDE (6) reduces to

$$\begin{aligned} &\alpha^3\beta\phi_{rrrr} - 6\alpha^2\beta\phi_r\phi_{rr} - \alpha\phi_{rr} + \beta\phi_{rr} + 4\alpha^3\beta\phi_{rsss} + 6\alpha^3\beta\phi_{rrss} + 4\alpha^3\beta\phi_{rrrs} \\ &- 12\alpha^2\beta\phi_s\phi_{rs} - 12\alpha^2\beta\phi_r\phi_{rs} - 6\alpha^2\beta\phi_{rr}\phi_s - 2\alpha\phi_{rs} + 2\beta\phi_{rs} + 2\phi_{rs} + \phi_{rr} \\ &+ \left(1 - \alpha + \beta - 6\alpha^2\beta\phi_r - 6\alpha^2\beta\phi_s\right)\phi_{ss} + \alpha^3\beta\phi_{ssss} = 0, \end{aligned} \tag{8}$$

which is PDE in two independent variables. Equation (8) has four Lie symmetries

$$\begin{aligned} &\Gamma_1 = \frac{\partial}{\partial \phi},\ \Gamma_2 = \frac{\partial}{\partial s},\ \Gamma_3 = \frac{\partial}{\partial r}, \\ &\Gamma_4 = \frac{1}{\alpha - \beta - 1}\left(\alpha r - \beta r - r + 3\alpha^2\beta\phi\right)\frac{\partial}{\partial \phi} - \frac{3\alpha^2\beta r}{\alpha - \beta - 1}\frac{\partial}{\partial r}. \end{aligned}$$

The symmetry $\Gamma = \gamma\Gamma_2 + \Gamma_3$ provides two invariants $\zeta = \phi$ and $p = r - \gamma s$. These invariants transform (8) into a nonlinear ODE

$$a\zeta'''' - b\zeta'\zeta'' + c\zeta'' = 0, \tag{9}$$

where $a = \alpha^3\beta\left(\gamma^4 - 4\gamma^3 + 6\gamma^2 - 4\gamma + 1\right)$, $b = 6\alpha^2\beta\left(1 - 3\gamma + 3\gamma^2 - \gamma^3\right)$,
$c = (1 - \alpha + \beta)(\gamma - 1)^2$ and $p = \alpha\beta\gamma t + \alpha(1 - \gamma)x + (\beta + 1)(\gamma - 1)y + \alpha\beta z$.

2.2 Solution of (2) using direct integration

In this subsection, we derive a solution of the (3+1)-dimensional negative-order KdV equation, model II (2) by direct integration of the ODE (9). Twice integration of (9) with respect to p gives

$$\frac{1}{2}a(\zeta'')^2 - \frac{1}{6}b(\zeta')^3 + \frac{1}{2}c(\zeta')^2 + C_1\zeta' + C_2 = 0 \tag{10}$$

with C_1, C_2 constants. Letting $\varphi = \zeta'$, equation (10) becomes

$$\varphi'^2 = \frac{b}{3a}\varphi^3 - \frac{c}{a}\varphi^2 - \frac{2C_1}{a}\varphi - \frac{2C_2}{a}. \tag{11}$$

Assume that r_1, r_2 and r_3 are roots of

$$\varphi^3 - \frac{3c}{b}\varphi^2 - \frac{6C_1}{b}\varphi - \frac{6C_2}{b} = 0$$

with $r_1 \geq r_2 \geq r_3$. Equation (11) now becomes

$$\varphi'^2 = \frac{b}{3a}(\varphi - r_1)(\varphi - r_2)(\varphi - r_3)$$

and its solution can be written in the form of Jacobi elliptic function (Kudryashov, 2004; Abramowitz & Stegun, 1972; Motsepa & Khalique, 2018)

$$\varphi(p) = r_2 + (r_1 - r_2)\mathrm{cn}^2\left\{\sqrt{\frac{b(r_1 - r_3)}{12a}}p, R^2\right\},\ R^2 = \frac{r_1 - r_2}{r_1 - r_3}, \tag{12}$$

where cn is the elliptic cosine function. Integrating equation (12) with respect to p and returning to original variables, we accomplish the solution of (3) as

$$\begin{aligned} v(x,y,z,t) = &\sqrt{\frac{12a\left(r_1 - r_2\right)^2}{b(r_1 - r_3)R^8}}\left\{\mathrm{EllipticE}\left[\mathrm{sn}\left(\sqrt{\frac{b(r_1 - r_3)}{12a}}p, R^2\right), R^2\right]\right\} \\ &+ \left\{r_2 - (r_1 - r_2)\,\frac{1 - R^4}{R^4}\right\} p + K, \end{aligned}$$

with $p = \alpha\beta\gamma t + \alpha(1-\gamma)x + (\beta+1)(\gamma-1)y + \alpha\beta z$, K a constant and $\mathrm{EllipticE}[q,k]$ the incomplete elliptic integral given by (Abramowitz & Stegun, 1972)

$$\mathrm{EllipticE}[q,k] = \int_0^q \sqrt{\frac{1 - k^2 s^2}{1 - s^2}}ds.$$

The solution of (2) is then given by differentiating v with respect to x. Thus,

$$\begin{aligned} u = &\alpha(1-\gamma)\frac{r_1 - r_2}{R^4}\,\mathrm{cn}\left(\sqrt{\frac{b(r_1 - r_3)}{12a}}p, R^2\right) \times \mathrm{dn}\left(\sqrt{\frac{b(r_1 - r_3)}{12a}}p, R^2\right) \\ &\times \sqrt{1 - R^2\sin^2\left[\mathrm{sn}\left(\sqrt{\frac{b(r_1 - r_3)}{12a}}p, R^2\right)\right]} \\ &+ \alpha(1-\gamma)\left(r_2 - \frac{\left(1 - R^4\right)(r_1 - r_2)}{R^4}\right). \end{aligned}$$

2.3 Solution of (2) using Kudryashov method

In this section, we employ the Kudryashov method (Kudryashov, 2012) and find exact solutions of the (3+1)-dimensional negative-order KdV equation, model II (2). This method is one of the most productive approaches for determining closed-form solutions of NPDEs. The first step is to reduce the NPDE (2) to nonlinear ODE, which we have already done using the Lie symmetries in the previous section. Thus, we work with the ODE (9). We assume the solution of (9) is of the form

$$\zeta(p) = \sum_{n=0}^{N} A_n H^n(p), \tag{13}$$

where $H(p)$ satisfies the first-order nonlinear ODE

$$H'(p) = H^2(p) - H(p). \tag{14}$$

We note that the solution of (14) is

$$H(p) = \frac{1}{1+\exp p}. \tag{15}$$

For equation (9), the balancing procedure yields $N = 1$. Thus, from (13), we have

$$\zeta(p) = A_0 + A_1 H(p). \tag{16}$$

Now substituting (16) into (9) and using (14), we obtain

$$\begin{aligned}
&H(p)^5 \left(-24\alpha^4 A_1\beta + 12\alpha^3 A_1^2\beta - 24\alpha^4 A_1\beta\gamma^4 + 96\alpha^4 A_1\beta\gamma^3 - 12\alpha^3 A_1^2\beta\gamma^3\right.\\
&-144\alpha^4 A_1\beta\gamma^2 + 36\alpha^3 A_1^2\beta\gamma^2 + 96\alpha^4 A_1\beta\gamma - 36\alpha^3 A_1^2\beta\gamma\big) + H(p)^4 \left(60\alpha^4 A_1\beta\right.\\
&-30\alpha^3 A_1^2\beta + 60\alpha^4 A_1\beta\gamma^4 - 240\alpha^4 A_1\beta\gamma^3 + 30\alpha^3 A_1^2\beta\gamma^3 + 360\alpha^4 A_1\beta\gamma^2\\
&-90\alpha^3 A_1^2\beta\gamma^2 - 240\alpha^4 A_1\beta\gamma + 90\alpha^3 A_1^2\beta\gamma\big) + H(p)\left(-\alpha^4 A_1\beta + \alpha^2 A_1 - \alpha A_1\beta\right.\\
&-\alpha A_1 - \alpha^4 A_1\beta\gamma^4 + 4\alpha^4 A_1\beta\gamma^3 - 6\alpha^4 A_1\beta\gamma^2 + \alpha^2 A_1\gamma^2 - \alpha A_1\beta\gamma^2 - \alpha A_1\gamma^2\\
&+4\alpha^4 A_1\beta\gamma - 2\alpha^2 A_1\gamma + 2\alpha A_1\beta\gamma + 2\alpha A_1\gamma\big) + H(p)^3\left(-50\alpha^4 A_1\beta + 24\alpha^3 A_1^2\beta\right.\\
&+2\alpha^2 A_1 - 2\alpha A_1\beta - 2\alpha A_1 - 50\alpha^4 A_1\beta\gamma^4 + 200\alpha^4 A_1\beta\gamma^3 - 24\alpha^3 A_1^2\beta\gamma^3\\
&-300\alpha^4 A_1\beta\gamma^2 + 72\alpha^3 A_1^2\beta\gamma^2 + 2\alpha^2 A_1\gamma^2 - 2\alpha A_1\beta\gamma^2 - 2\alpha A_1\gamma^2 + 200\alpha^4 A_1\beta\gamma\\
&-72\alpha^3 A_1^2\beta\gamma - 4\alpha^2 A_1\gamma + 4\alpha A_1\beta\gamma + 4\alpha A_1\gamma\big) + H(p)^2\left(15\alpha^4 A_1\beta - 6\alpha^3 A_1^2\beta\right.\\
&-3\alpha^2 A_1 + 3\alpha A_1\beta + 3\alpha A_1 + 15\alpha^4 A_1\beta\gamma^4 - 60\alpha^4 A_1\beta\gamma^3 + 6\alpha^3 A_1^2\beta\gamma^3\\
&+90\alpha^4 A_1\beta\gamma^2 - 18\alpha^3 A_1^2\beta\gamma^2 - 3\alpha^2 A_1\gamma^2 + 3\alpha A_1\beta\gamma^2 + 3\alpha A_1\gamma^2 - 60\alpha^4 A_1\beta\gamma\\
&+18\alpha^3 A_1^2\beta\gamma + 6\alpha^2 A_1\gamma - 6\alpha A_1\beta\gamma - 6\alpha A_1\gamma\big) = 0.
\end{aligned}$$

Splitting on the powers of $H(p)$, yields the following algebraic equations for the coefficients A_0 and A_1:

$$\begin{aligned}
&12\alpha^3 A_1^2\beta - 24\alpha^4 A_1\beta - 24\alpha^4 A_1\beta\gamma^4 + 96\alpha^4 A_1\beta\gamma^3 - 12\alpha^3 A_1^2\beta\gamma^3\\
&- 144\alpha^4 A_1\beta\gamma^2 + 36\alpha^3 A_1^2\beta\gamma^2 + 96\alpha^4 A_1\beta\gamma - 36\alpha^3 A_1^2\beta\gamma = 0,\\
&60\alpha^4 A_1\beta - 30\alpha^3 A_1^2\beta + 60\alpha^4 A_1\beta\gamma^4 - 240\alpha^4 A_1\beta\gamma^3 + 30\alpha^3 A_1^2\beta\gamma^3\\
&+ 360\alpha^4 A_1\beta\gamma^2 - 90\alpha^3 A_1^2\beta\gamma^2 - 240\alpha^4 A_1\beta\gamma + 90\alpha^3 A_1^2\beta\gamma = 0,\\
&\alpha^2 A_1 - \alpha^4 A_1\beta - \alpha A_1\beta - \alpha A_1 - \alpha^4 A_1\beta\gamma^4 + 4\alpha^4 A_1\beta\gamma^3 - 6\alpha^4 A_1\beta\gamma^2\\
&+ \alpha^2 A_1\gamma^2 - \alpha A_1\beta\gamma^2 - \alpha A_1\gamma^2 + 4\alpha^4 A_1\beta\gamma - 2\alpha^2 A_1\gamma + 2\alpha A_1\beta\gamma + 2\alpha A_1\gamma = 0,\\
&24\alpha^3 A_1^2\beta - 50\alpha^4 A_1\beta + 2\alpha^2 A_1 - 2\alpha A_1\beta - 2\alpha A_1 - 50\alpha^4 A_1\beta\gamma^4 + 200\alpha^4 A_1\beta\gamma^3\\
&- 300\alpha^4 A_1\beta\gamma^2 + 72\alpha^3 A_1^2\beta\gamma^2 + 2\alpha^2 A_1\gamma^2 - 2\alpha A_1\beta\gamma^2 - 2\alpha A_1\gamma^2 + 200\alpha^4 A_1\beta\gamma\\
&- 72\alpha^3 A_1^2\beta\gamma - 4\alpha^2 A_1\gamma + 4\alpha A_1\beta\gamma + 4\alpha A_1\gamma - 24\alpha^3 A_1^2\beta\gamma^3 = 0,\\
&15\alpha^4 A_1\beta - 6\alpha^3 A_1^2\beta - 3\alpha^2 A_1 + 3\alpha A_1\beta + 3\alpha A_1 + 15\alpha^4 A_1\beta\gamma^4 - 60\alpha^4 A_1\beta\gamma^3\\
&+ 6\alpha^3 A_1^2\beta\gamma^3 + 90\alpha^4 A_1\beta\gamma^2 - 18\alpha^3 A_1^2\beta\gamma^2 - 3\alpha^2 A_1\gamma^2 + 3\alpha A_1\beta\gamma^2 + 3\alpha A_1\gamma^2\\
&- 60\alpha^4 A_1\beta\gamma + 18\alpha^3 A_1^2\beta\gamma + 6\alpha^2 A_1\gamma - 6\alpha A_1\beta\gamma - 6\alpha A_1\gamma = 0.
\end{aligned}$$

The solution of the above algebraic system, using Mathematica, is

$$A_0 = A_0,\ A_1 = 2\alpha\left(1-\gamma\right),\ \gamma = \frac{\alpha^3\beta - \sqrt{\alpha^3\beta(\alpha-\beta-1)}}{\alpha^3\beta}.$$

Thus, the solution of (3) is given by

$$v(x,y,z,t) = A_0 + \frac{A_1}{1+\exp p} \tag{17}$$

and consequently using the fact that $u = v_x$, the solution of (2) is obtained by differentiating (17) with respect to x, which is

$$u(x,y,z,t) = -\frac{\alpha(1-\gamma)A_1 \exp\{\alpha\beta\gamma t + \alpha(1-\gamma)x + (\beta+1)(\gamma-1)y + \alpha\beta z\}}{[1+\exp\{\alpha\beta\gamma t + \alpha(1-\gamma)x + (\beta+1)(\gamma-1)y + \alpha\beta z\}]^2}.$$

3 Conservation laws of (2)

In this section, we construct conservation laws for the (3+1)-dimensional negative-order KdV equation, model II (2). We employ Noether's theorem.

As seen in Section 2.1, equation (3) consists of nine Lie symmetries given by (4). Since the variational symmetries are a subset of Lie point symmetries of a given PDE, using the results in Anco (2017), we conclude that only X_1, X_3, X_4, X_5 and X_6 are found to be variational symmetries. It can be verified that a Lagrangian of equation (3) is

$$\mathcal{L} = v_x^2 v_t - v_x^2 v_z + \frac{1}{2}\left(v_{xx}v_{xt} - v_{xx}v_{xz} - v_x^2 - v_x v_y\right). \tag{18}$$

Thus, the conserved vectors corresponding to the Noether point symmetries X_1, X_3, X_4, X_5 and X_6, using Sarlet (2010)

$$T^k = Ł\xi^k + \left(\eta^\alpha - u^\alpha_{x^j}\xi^j\right)\left(\frac{\partial Ł}{\partial u^\alpha_{x^k}} - \sum_{l=1}^{k} D_{x^l}\left(\frac{\partial Ł}{\partial u^\alpha_{x^l x^k}}\right)\right) + \sum_{l=k}^{n}\left(\zeta^\alpha_l - u^\alpha_{x^l x^j}\xi^j\right)\frac{\partial Ł}{\partial u^\alpha_{x^k x^l}}$$

are given by, respectively

$$\begin{aligned}
T_1^x = {} & F^1(t+z)\left(\frac{1}{2}u_y u_z + 2u\int u_y\,dx \int (u_z - u_t)\,dx + u\int u_y\,dx + \frac{1}{4}u_x\int u_{yz}\,dx\right.\\
& \left. + \frac{3}{4}u_{xt}\int u_y\,dx - \frac{1}{4}u_x\int u_{yt}\,dx - \frac{1}{2}u_t u_y - \frac{3}{4}u_{xz}\int u_y\,dx + \frac{1}{2}\left(\int u_y\,dx\right)^2\right),
\end{aligned}$$

$$\begin{aligned}
T_1^y &= F^1(t+z)\left(u^2\int u_t\,dx - u^2\int u_z\,dx - \frac{1}{2}u^2 + \frac{1}{2}u_t u_x - \frac{1}{2}u_x u_z\right),\\
T_1^z &= F^1(t+z)\left(u^2\int u_y\,dx + \frac{1}{4}u_x u_y - \frac{1}{4}u_{xx}\int u_y\,dx\right),\\
T_1^t &= F^1(t+z)\left(\frac{1}{4}u_{xx}\int u_y\,dx - u^2\int u_y\,dx - \frac{1}{4}u_x u_y\right);
\end{aligned}$$

$$\begin{aligned}
T_3^x &= \frac{1}{2}F^3_y v + F^3(y,t+z)\left(2u\int (u_t - u_z)\,dx - u - \frac{3}{4}u_{xt} - \frac{1}{2}\int u_y\,dx + \frac{3}{4}u_{xz}\right),\\
T_3^y &= -\frac{1}{2}F^3(y,t+z)u,\\
T_3^z &= \frac{1}{4}F^3(y,t+z)\left(u_{xx} - 4u^2\right),\\
T_3^t &= F^3(y,t+z)\left(u^2 - \frac{u_{xx}}{4}\right);
\end{aligned}$$

$$T_4^x = F^4(t+z)\left(2u\left(\int u_t\,dx\right)^2 - 4u\int u_t\,dx\int u_z\,dx - u\int u_t\,dx + 2u\left(\int u_z\,dx\right)^2\right.$$

$$+u\int u_z\,dx-\frac{1}{2}\int u_t\,dx\int u_y\,dx+\frac{3}{4}u_{xz}\int u_t\,dx-\frac{1}{2}u_x\int u_{zt}\,dx$$
$$+\frac{3}{4}u_{xt}\int u_z\,dx-\frac{3}{4}u_{xt}\int u_t\,dx+\frac{1}{4}u_x\int u_{tt}\,dx-u_tu_z+\frac{1}{2}u_t^2$$
$$+\frac{1}{2}\int u_y\,dx\int u_z\,dx+\frac{1}{4}u_x\int u_{zz}\,dx-\frac{3}{4}u_{xz}\int u_z\,dx+\frac{u_z^2}{2}\Bigg),$$

$$T_4^y=\frac{1}{2}uF^4(t+z)\left(\int u_z\,dx-\int u_t\,dx\right),$$
$$T_4^z=\frac{1}{4}F^4(t+z)\left(u_tu_x-u_xu_z-2u\int u_y\,dx-2u^2+u_{xx}\int\left(u_t-u_z\right)\,dx\right),$$
$$T_4^t=\frac{1}{4}F^4(t+z)\left(2u\int u_y\,dx+2u^2-u_tu_x-u_{xx}\int u_t\,dx+u_xu_z+u_{xx}\int u_z\,dx\right);$$

$$T_5^x=F^5(y,t+z)\left(\frac{3}{2}u_{xt}u-\frac{3}{2}u_{xz}u-\frac{1}{2}u_tu_x+\frac{1}{2}u_xu_z+2u^2\int\left(u_z-u_t\right)dx+u^2\right)$$
$$+F_y^5\left(2zu\int\left(u_t-u_z\right)dx-zu-\frac{3}{4}zu_{xt}+\frac{3}{4}zu_{xz}-\frac{1}{2}z\int u_y\,dx-\frac{1}{4}u_x\right)$$
$$+\frac{1}{2}zF_{yy}^5\int u\,dx,$$
$$T_5^y=F^5(y,t+z)u^2-\frac{1}{2}zF_y^5u,$$

$$T_5^z=F^5(y,t+z)\left(\frac{1}{2}u_x^2+2u^3-\frac{1}{2}u_{xx}u\right)-zF_y^5u^2+\frac{1}{4}zu_{xx}F_y^5,$$
$$T_5^t=F^5(y,t+z)\left(\frac{1}{2}u_{xx}u-\frac{1}{2}u_x^2-2u^3\right)+zF_y^5u^2-\frac{1}{4}zu_{xx}F_y^5;$$

$$T_6^x=F^6(t+z)\Bigg(4yu\int u_t\,dx+16yu\int u_t\,dx\int u_z\,dx-8yu\left(\int u_t\,dx\right)^2-2xu\int u_t\,dx$$
$$-\frac{3}{4}xu_{xz}-2yu_z^2+\frac{1}{2}u_z+2xu\int u_z\,dx-4yu\int u_z\,dx+xu-3yu_{xz}\int u_t\,dx$$
$$+2yu_x\int u_{zt}\,dx-3yu_{xt}\int u_z\,dx+2y\int u_t\,dx\int u_y\,dx+3yu_{xt}\int u_t\,dx$$
$$-yu_x\int u_{tt}\,dx+\frac{3}{4}xu_{xt}+4yu_tu_z-2yu_t^2-\frac{u_t}{2}-2y\int u_y\,dx\int u_z\,dx-\int u\,dx$$
$$-yu_x\int u_{zz}\,dx+3yu_{xz}\int u_z\,dx+\frac{1}{2}x\int u_y\,dx-8yu\left(\int u_z\,dx\right)^2\Bigg),$$
$$T_6^y=F^6(t+z)\left(2yu\int u_t\,dx-2yu\int u_z\,dx+\frac{1}{2}xu-\frac{1}{2}\int u\,dx\right),$$
$$T_6^z=F^6(t+z)\Bigg(xu^2+2yu^2-yu_tu_x-\frac{1}{4}xu_{xx}+yu_xu_z+\frac{1}{4}u_x+yu_{xx}\int u_z\,dx$$
$$+2yu\int u_y\,dx-yu_{xx}\int u_t\,dx\Bigg),$$
$$T_6^t=F^6(t+z)\Bigg(yu_tu_x-2yu\int u_y\,dx-xu^2-2yu^2+yu_{xx}\int u_t\,dx-yu_xu_z-\frac{1}{4}u_x$$
$$-yu_{xx}\int u_z\,dx+\frac{1}{4}xu_{xx}\Bigg).$$

4 Concluding remarks

In this paper, we studied the (3+1)-dimensional negative-order Korteweg-de Vries equation (2). This equation was one of the higher-dimensional models, which was recently formulated by using the recursion operator of the Korteweg-de Vries equation. The integral appearing in the equation was first eliminated to obtain a fourth-order nonlinear partial differential equation. Lie symmetries were computed for this fourth-order partial differential equation and the process of symmetry reductions produced a fourth-order nonlinear ordinary differential equation whose closed-form solutions were then obtained. Furthermore, conserved vectors were derived for the underlying equation by invoking Noether's theorem.

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